A Comparison of Two Estimators for Solutions to Greedy Algorithm in Scheduling Depletable Sources

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Abstract: There are many depletable resource planning problems associated with convex cost function. The Greedy algorithm has been used to find the optimal solution for such problems. But exploiting that algorithm requires a large bulk of computations to find the optimal solution. Thus, in this paper we have established two piecewise linear estimators for the convex cost function: an inner intersection of the function values and an outer tangent. A practical experiment was conducted to compare the performance of the two presented estimators and it was observed that the outer tangent performs more efficiently than the inner intersection. The execution time of the Greedy Algorithm and the estimation algorithm in this paper have been compared, and the results reveal that our algorithm works 54 times faster than the Greedy Algorithm.

Keywords; Convex cost function, greedy algorithm, depletable sources, linear estimation.

1 INTRODUCTION

It is assumed in many resources planning problems that the extraction cost for exhaustible energy sources is convex, and distribution and inventory related cost are linear (Modiano et al., 1980). This results in a total convex supply cost. Thus, there is a need for a technique to solve a model consist of convex cost. The computational complexity of these problems has been taken into account and several algorithms have been proposed for solving them (Magnanti et al., 2006).

Minimizing transportation or supply cost functions often involves substituting the non-linear function by a piecewise linear estimation (Devine et al., 1972). In order to approximate more precisely, we need to increase the number of pieces. A traditional example of such an algorithm is the convex cost flow scaling algorithm (Ahuja et al., 1993).

Nonlinear convex cost functions are considered to be modeled as a piecewise linear function various contexts. In this framework, resource constraints are specified to accommodate the possibility that the usage of shared resources depends on the production quantity by source or cost segment. This model was formulated by Bowman as a transportation problem, where there are multiple periods and multiple production option, but only one item and one resource type (Graves, 1999). The transportation problem is an essential problem in optimization and is well known that it is polynomially solvable even when the flows are required to be integers (Bachelet et al., 2003, Kuno et al.,2007 and Arkin et al., 2004). The demand constraints in the stochastic transportation problem, have been replaced by nonlinear convex costs as functions of the total inflow into each demand point (Cooper et al., 1997 and Holmberg, 1984).

Convex piecewise linear functions are in some cases used to estimate a combination of discontinuous cost functions. Salman et al. (2007) conducted an experiment in which cable costs were closely approximated by the piecewise-linear convex hull, using SOR (Successive Over relaxation Method).

Convex cost minimization problems which are obtained from combined model networks, such as a two mode (private car and public transit) traffic model (Florian and Spiess 1983) can be efficiently solved by adaptation of the linear approximation method (Hall, 1999). A combined form in which the arc costs are piecewise convex, with decreasing derivatives at breakpoints is considered in Mahey et al., (2007).

Modiano et al. (1980), proposed a dynamic model that determines the economic supply for a depletable source. They employed a linear programming model to analyze energy sources such as gas, oil, and coal in energy producing divisions. Stoecker et al. (1985) introduced an effective technique for irrigation systems investment planning. The corresponding method utilizes linear and also dynamic programming as means for achieving the maximum future returns.

A water recycling problem is demonstrated by Hartl et al. (2006). Using linear programming methodology, they developed a model which enables them to determine the optimal use of configuration pumps. Moreover, in this method it is possible to obtain the solution through sorting based on cost elements.

In this paper we introduce and compare two convex piecewise-linear estimators for the convex supply cost: an inner intersection and an outer tangent to the convex cost function. We show that are proposed algorithm, compared with the Greedy Algorithm or also known as the Convex Cost Algorithm (Johnson et al., 1972), provides a far more efficient way to solve problems associated with convex cost. Our approximation approach, while being almost as precise as the Greedy Algorithm, requires less computation and execution time

The paper is organized as follows. The next section presents the depletable resource model which is associated with convex extraction cost, and thus overall convex cost. The third section introduces our piecewise linear estimation approach, and the algorithm for creating a computer program for this purpose. Two estimators for the convex cost function are presented in this section. In the forth section, we have conducted a practical experiment to demonstrate the results provided by our piecewise linear estimation of the convex cost. Some relevant analysis are also provided in this section. The paper is concluded in the last section.

2 STATEMENT OF MODEL

The model utilized in this paper depicts a scheduling policy in order to incur demands for depletable commodities over a finite planning horizon. The process of supplying these commodities is associated with costs of supply, conversion and distribution. For each period $t = 1,...,T$, we aim to schedule the supply time of demands in a way to optimize costs as well.

The following notation is needed:

$r_t =$ supply scheduled for period t $(t = 1,2,...,T)$

$d_t =$ expected demand in period t

$i_t =$ net inventory at the end of period t

$K_t(r_t, i_t) =$ cost of producing r_t units and having an ending net inventory of i_t in period t

$e(r_t) =$ accumulative extraction cost

$f(r_t) =$ other related costs of conversion and distribution

$h(r_t) =$ inventory costs

$g(r_t) =$ total supply cost

Considering the resource planning point of view, it is assumed the inventory related costs depend only on the net inventory and the supply costs depend only on the supply rate. Thus we can write:

$$K_t(r_t, i_t) = e(r_t) + f(r_t) + h(i_t) \quad (1)$$

The extraction cost function $e(r_t)$ for the depletable source is assumed to be convex and increasing, and $f(r_t)$ is a linear function of supply. Thus, we define the overall cost of supply by the following equation.

$$g(r_t) = e(r_t) + f(r_t) \quad (2)$$

The total cost, $g(r_t)$, is also convex regarding supply of depletable sources. Also, we suppose that shortages are backlogged, thus i_t can take on negative values as well as positive values. The objective of this linear programming problem is therefore supply planning in order to incur demands of each time period *t*, and minimize overall cost.

The Greedy algorithm can be utilized to schedule supply over a finite planning horizon. This algorithm suggests to consider the *T* possible alternatives for the production time of each unit of demand in each period. The entire incremental cost from extraction, conversion, distribution, inventory and backlogging is then computed and supply plan is adjusted to the alternative with the minimum cost. This process is continued for all units of demand in each period. This algorithm can be employed in cases with a convex supply cost function. However, implementing this algorithm, in many cases requires burdensome computation whereas minimizing numerical computations might be a substantial objective in many cases.

On the other hand, for piecewise linear cost functions, the transportation tableau (Taha, 1982), can be used in the context of resource planning when a problem has a convex piecewise-linear supply cost and linear inventory holding and backlogging costs. When backorders are allowed, the transportation algorithm is a more efficient way for solving the problem, than the convex cost algorithm. However, when backlogging is not permitted, convex cost algorithm swiftly provides the solution (Johnson, 1974).

The transportation model can be portrayed by a network with *m* sources and *n* destinations. A source or a destination is represented by a node. The arc joining a source and a destination represents the route through which the commodity is transported. The amount of supply at source i is a_i and the demand at destination j is b_j. The unit transportation cost between source I and destination j is c_{ij}. Let x_{ij} represent the transported amount from source i to destination j; then the LP model representing the transportation problem is given generally as:

$$\text{Minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \quad (3)$$

Subject to

$$\sum_{j=1}^{n} x_{ij} \le a_i, \quad i = 1,2,\dots,m \quad (4)$$

$$\sum_{i=1}^{m} x_{ij} \ge b_j, \quad j = 1,2,\dots,n \quad (5)$$

$$x_{ij} \ge 0 \quad \text{for all } i \text{ and } j \quad (6)$$

A more compact way to represent the transportation model is to use the transportation tableau which is a matrix form with its rows representing the sources and its columns the destination. The cost elements c_{ij} are summarized in the northeast corner of the matrix cell (i, j) (Taha, 1982).

The following procedure shows the required steps for solving a dynamic programming model which has a transportation model form.

1. Satisfy demand in the first period by the least cost source.
2. Adjust capacities to indicate amounts remaining for step1.
3. Satisfy demand in the second period by the least-cost sources.
4. Adjust capacities to indicate amounts remaining after step3.
5. Repeat steps 3 and 4, for periods 3,4,…,T (Johnson, 1974)

In the next section, we present an algorithm for estimating two approximations for a convex cost function in which the concept of transportation is employed to minimize the cost and omit the large bulk of computations of the greedy algorithm.

3 OUR APPROXIMATION APPROACH

For the convenience of the transportation algorithm especially when the supply cost is a convex non-linear function, we estimate a non-linear convex supply cost function with a convex piecewise-linear function. Thus, the problem can be solved more efficiently by the transportation algorithm rather than the convex cost algorithm.

In this paper we estimate a convex function using two piecewise-linear functions as an inner intersection, and an outer tangent to the convex function. We then solve the model via the transportation algorithm. If these two functions are chosen accurate enough, then the results are very close to that of a convex cost algorithm. The accuracy is relevant to the length of the increment we choose for dividing the supply (r) axis into even pieces.

Consider the supply and inventory cost function. Here, we assume that the supply cost, $g(r)$ is convex, and the net inventory cost is linear, thus $K_t(r_t, i_t)$ would be convex as a whole. Moreover, the net inventory cost is separated into two linear functions which represent the inventory carrying cost $h_t^+(i_t)$ and the backorder cost $h_t^-(i_t)$ as shown in the following formulas.

$$K_t(r_t, i_t) = g_t(r_t) + h^+{}_t(i_t) + h_t^-(i_t) \quad (7)$$

$$h_t^+(i_t) = \lambda \cdot i \quad (8)$$

$$h_t^-(i_t) = \pi \cdot i \quad (9)$$

As previously mentioned, we use two piecewise-linear functions, as the inner intersection and outer tangent to estimate the non-linear convex supply cost. In order to do so, we should choose an increment for the supply and divide the supply (r) axis into even pieces. Then we can obtain the inner intersection by joining the function values in the break points. The outer tangent however, can be reached by drawing tangent lines on the curve, in the break points. Figure 1 illustrates the convex supply cost function and the two piecewise-linear estimators. The piecewise-linear function, g_1, is the inner intersection and , g_2, is the outer tangent to the non-linear convex function $g_t(r_t)$.

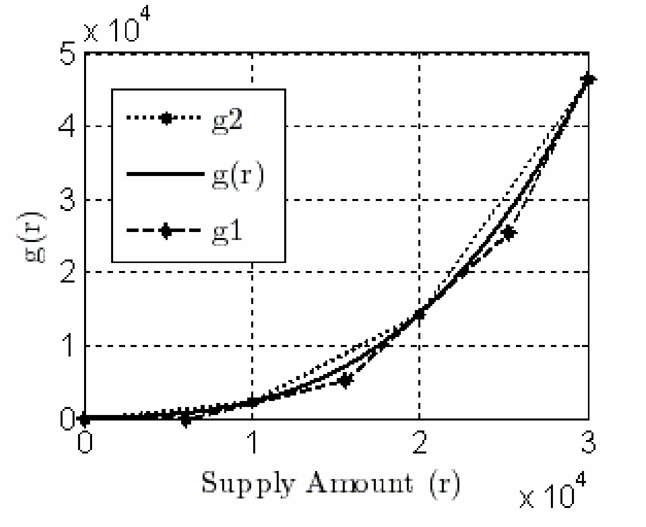

Figure 1. The convex supply cost function and the two piecewise linear estimators.

The stepwise algorithm has been summarized in the figure below for estimating the two approximations of the convex cost function.

Inputs

Number of periods: T
Demands of each period: $d_t = [d_1 .... d_T]$
Convex cost function $g(r)$
Backorder costs: c
Inventory of each period $i_t = [i_1,...,i_T]$

Begin

Compute average and maximum of demands: $\langle d_t \rangle$ and $d_{\max}$.
Initialize supply increment as $\Delta r = 2^{\lfloor \log_2 \langle d_t \rangle \rfloor}$
Assume supply range $0 \le r \le r_f$ such that $\exists n \in N$, $r_f = n \times \Delta r \ge d_{\max}$.
Assume a tolerance of estimation error: *tol*.

While $r_2 - r_1 > tol$

Assume piecewise supply range: $r_1 = [0, \Delta r, ..., r_f]$.
Compute piecewise cost $g_1 = g(r_1)$ and marginal costs m_{C1}.
Satisfy d_t with a transportation tableau of marginal costs m_{C1} and capacities equal to $\Delta r + 1$ for each supply Interval, considering i_t and h_t.
Compute the total cost K_1.
Compute tangents to $g(r)$ at each break point. $r_1 = [0, \Delta r, ..., r_f]$
Find new set of break points r_2 and corresponding g_2 located at incidents of tangents.
Satisfy d_t with a transportation tableau of marginal costs m_{C2} ..
Compute the total cost K_2.
Set new increment to $\Delta r = \Delta r / 2$.

End
End

Figure 2. Our estimation algorithm

4 PRACTICAL EXPERIMENT

In order to justify the performance of the presented algorithm, an experiment was conducted for a cost function which was obtained by fitting a quadratic polynomial to the data provided by Sangrud coal mine in Iran for the year 2005. The cost function is convex because its marginal values are increasing. The following equation demonstrates the corresponding convex cost function which attains cost in $ 10^6 for input supply amount in (10^{20} Joules) to be scheduled in a six month period T=6.

$$Y_g(r) = 9.8e^{-4}(r^2) + 0.62(r) + 0.5 \quad (10)$$

In order to evaluate the performance of the algorithm, the corresponding total costs for the two estimators which are piecewise linear cost functions g_1 and g_2 were estimated. The estimation of overall costs K_1 and K_2 via the two estimators was done using the algorithm elaborated in Figure 2. The experiment evaluates the algorithm for the cost function, $Y_g(r)$, for all 6 periods, $t = 1,...,6$.

The following suppositions are to be considered regarding the experiment.

- The demands to be satisfied were chosen as $d_t = [10, 35, 25, 15, 17, 28]$.
- The initial inventory was set to zero.
- Backorder costs were assumed linear and equal to $h_t = [3,2,1,1,1].*0.3*\max\{\mathrm{m_C}\}$ where m_C indicates marginal cost for estimated piecewise linear cost function.
- The amount of supply increment Δr was varied from initialized value (refer to Figure 3.) to 1. The variation of K_1 and K_2 is depicted in Figure 3. with a logarithmic scale of $\log_2$.

It can be observed from the graph that by decreasing the increment value, the two estimators incline to the actual convex function. However, it can be obtained that K_2 is a better estimator because, it inclines to the actual cost function value more swiftly than K_1 . A more precise measure such as the "relevant error" could be used to demonstrate this statement. When comparing the corresponding factor of these two estimators, we discover that the relevant error of K_2 diminishes considerably faster that that of K_1 . Thus, we can conclude that the estimator, K_2 , which was obtained by outer tangents of the convex cost function, gives a better approximation for the convex cost function.

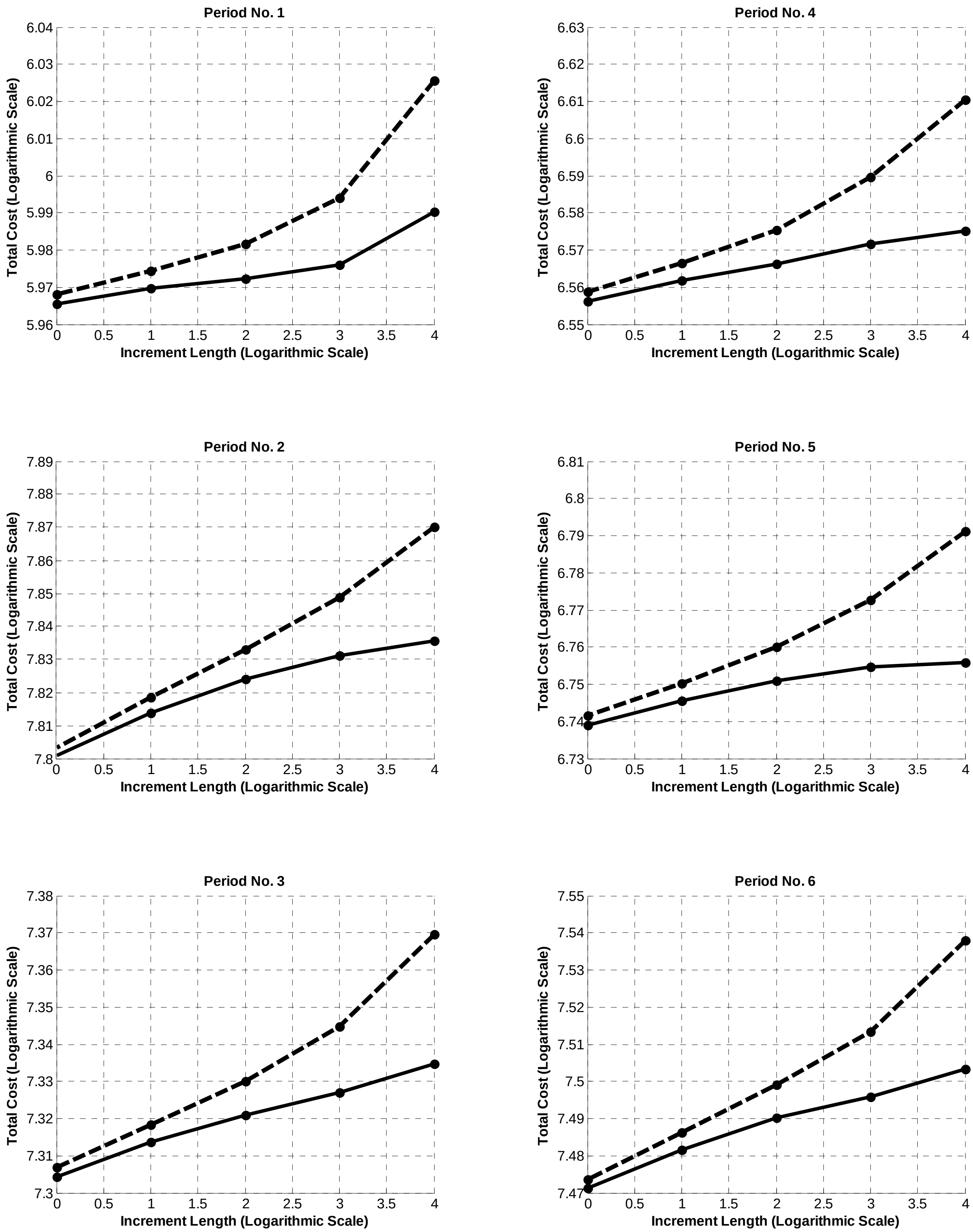

Figure 3. Two estimations of total cost for the cost function $Y_g(r)$: The solid line indicates intersection estimation (K_1) and the dashed line shows tangent estimation (K_2) for the convex cost function.

A reason for the efficiency of K_2 in comparison with K_1 could be that the slopes of K_2 are less steep than that of K_1. As we realized in the procedure of the transportation tableau, the demands are satisfied base on their marginal costs. The marginal costs are formed in with an increasing order. Thus, first the demands with smaller costs are met. In other words, the marginal costs with less steep slopes are first satisfies. So as we can see from the graph in Figure 3. the marginal cost for estimator K_2 are less steep and thus would be incurred earlier than that of K_1. This causes K_2 to be able to dampen its relative error more quickly and reach theactual convex cost function value.

It is interesting to note that the execution time for our estimation algorithm, using MATLAB software with normal CPU, was 0.13 sec. for one increment. However, when we made a computer program for the convex cost algorithm under the same conditions, we observed that the execution time was 7.02 sec. Thus, we can observe that our proposed algorithm considerably minimizes the large amount of computations associated with the greedy algorithm.

5 CONCLUSION

In this paper we introduced a method for approximating a convex supply cost function with two convex piecewise-linear functions. One of these piecewise linear estimators represents the inner intersection and the other one portrays the outer tangent of the function. Moreover, it was observed that the two estimators incline to the main convex function by decreasing the length of the production increment.

The experiment showed that the outer tangent, K_2 gives a better estimation of the convex cost function than the inner intersection, K_1. The reason could be that the slopes of K_2 are less steep than that of K_1. As we realized in the procedure of the transportation tableau, the demands are satisfied base on their marginal costs. The marginal costs are formed in with an increasing order. Thus, first the demands with smaller costs are met. In other words, the marginal costs with less steep slopes are first satisfies. So as we can see from the graph in Figure 3. that the marginal cost for estimator K_2 are less steep and thus would be incurred earlier than that of K_1.

The advantage of the proposed algorithm is omitting a large bulk of computations of the Convex Cost Algorithm, which has not been examined adequately in previous literature. Our experiment revealed that the estimation algorithm presented in this paper, works 54 times faster than the Greedy Algorithm (Convex Cost Algorithm). The defined estimators provide two approximations to the corresponding convex functions. Moreover, a desired precision can be obtained by adjusting the algorithm parameters.

As for future research, we recommend a more precise mathematical proof of the algorithm which is proposed by the paper and its numerical solutions are provided. Thus further work could be concentrated on a mathematical representation of inclination of two total cost functions as approximations to the convex cost function. Also, it can be shown that the approximation error decreases with tightening the increment length, specially for the outer tangent, K_2.

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